# Linearly ordered Splitting families

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## Introduction

Let S and X be infinite subsets of  $\omega$ . We say that S splits X if  $S \cap X$  and  $X \setminus S$  are both infinite. A family  $S \subseteq [\omega]^{\omega}$  is called an *splitting family* if for every  $X \in [\omega]^{\omega}$  there is  $S \in S$  such that S splits X. We will say a family is *linearly ordered* if it is linearly ordered under the almost inclusion (recall that A is an *almost subset of B* (denoted by  $A \subseteq^* B$ ) if  $A \setminus B$  is finite).

#### Problem

Are there linearly ordered splitting families?

Note that if  $\mathcal{S}$  is linearly ordered splitting family then:

- $\bigcirc$  S does not have a smallest or largest element.
- There are no inmmediate succesors, in particular it can not be a well order.

We can construct such families assuming the Continuum Hypothesis.

### Definition

Let  $\kappa$  and  $\lambda$  be two regular cardinal numbers. We say  $\mathcal{G} = (\mathcal{A}, \mathcal{B})$  is a  $(\kappa, \lambda)$ -pregap if the following holds:

- **(**  $\mathcal{A}$  is an increasing family (under the almost inclusion) of size  $\kappa$ .
- **2**  $\mathcal{B}$  is a decreasing family (under the almost inclusion) of size  $\lambda$ .
- If  $A \in A$  and  $B \in B$  then  $A \subseteq^* B$ .

A pregap  $\mathcal{G} = (\mathcal{A}, \mathcal{B})$  is a *gap* if it *can not be filled* (i.e. there is no  $X \in [\omega]^{\omega}$  such that  $A \subseteq^* X \subseteq^* B$  for every  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ ). The construction (under CH) of a linearly ordered splitting family can be easily done with the following result:

### Lemma (Rothberger, Hausdorff)

There are no  $(\kappa, \lambda)$ -gaps where  $\kappa, \lambda \in \{0, 1, \omega\}$ .

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Antonio Aviles and Felix Cabello constructed interesting Banach spaces assuming the existence of a linearly ordered splitting family. This lead them to ask the following:

### Problem (Aviles, Cabello)

*Is the existence of a linearly ordered splitting family consistent with the failure of* CH?

Frequently, a "CH construction" can be realized assuming that certain cardinal invariant is equal to c (the cardinality of  $2^{\omega}$ ). In this case the natural cardinal invariant would be the following:

#### Definition

Let j be the least  $\kappa$  for which there is a  $(\kappa, \kappa)$ -gap.

Obviously, a linearly ordered splitting family can be constructed assuming  $\mathfrak{j}=\mathfrak{c}.$  However we did not get anything new:

### Theorem (Hausdorff)

There is a  $(\omega_1, \omega_1)$ -gap (i.e.  $\mathfrak{j} = \omega_1$ )

Hence, a straightforward generalization of the previous argument can not be done if  $\mathfrak{c}>\omega_1.$ 

A natural attempt to solve the problem, would be to construct a Sacks indestructible linearly ordered splitting family. However, this idea is also doom to fail because of the following result:

#### Theorem

Every linearly ordered splitting family has size continuum.

#### Definitions

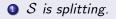
- Let  ${\mathcal S}$  be a linearly ordered family and  ${\mathcal G}=({\mathcal A},{\mathcal B})$  a pregap.
  - We say G is a *tight pregap* if there is no X ∈ [ω]<sup>ω</sup> such that the following holds:
    - $X \subseteq^* B$  for every  $B \in \mathcal{B}$ .
    - **2**  $X \cap A$  is finite for every  $A \in \mathcal{A}$ .

2 We say  $\mathcal{G}$  is a *cut of*  $\mathcal{S}$  if ( $\mathcal{G}$  is a pregap) and  $\mathcal{S} = \mathcal{A} \cup \mathcal{B}$ .

### We can then prove the following:

#### Lemma

Let S be a linearly ordered family. The following are equivalent:



2 Every cut of S is a non tight pregap.

#### Theorem

There is a linearly ordered splitting family in the Cohen model.

To prove the previous result we need the following definition:

#### Lemma

Let  $\mathcal{G} = \langle \mathcal{A}, \mathcal{B} \rangle$  be a pregap. We define the forcing  $\mathbb{P}(\mathcal{G})$  as the set of all  $p = (s_p, L_p, R_p)$  where  $s_p \in [\omega]^{<\omega}$ ,  $L_p \in [\mathcal{A}]^{<\omega}$ ,  $R_p \in [\mathcal{B}]^{<\omega}$  and  $\Delta(L_p, R_p) = \{\Delta(\mathcal{A}, \mathcal{B}) \mid \mathcal{A} \in L_p \land \mathcal{B} \in R_p\} \subseteq \max(s_p)$ . If  $p, q \in \mathbb{P}(\mathcal{A}, \mathcal{B})$  then  $p \leq q$  if the following holds: **1**  $s_q \sqsubseteq s_p, L_q \subseteq L_p, R_q \subseteq R_p$ . **2** If  $\max(s_q) < i \leq \max(s_p)$  then: **3** If  $i \in \bigcup L_q$  then  $i \in s_p$ . **3** If  $i \notin \bigcap R_q$  then  $i \notin s_p$ . Regarding the non existence we have the following:

Theorem

 $\mathsf{OCA} + \mathfrak{p} > \omega_1$  implies that there are no linearly ordered splitting families.

We know that  $\mathfrak{p} > \omega_1$  is not enough for the following result, but we do not know if OCA suffies to destroy such families.

### Problem

Does OCA implies that there are no linearly ordered splitting families?